II. Computability over the Reals

- real numbers: binary vs. approximate
- sequences, limits, rate of convergence
- function computability and continuity
- real arithmetic, join, maxim., integral
- root finding, argmax, derivative
- uncomputable wave equation
- analytic functions, discrete enrichment
- multivaluedness/non-extensionality: computability in linear algebra

Computable Real Numbers

Theorem: For $r\in\mathbb{R}$, the following are equivalent:

Call $r\in\mathbb{R}$ computable if

There is an algorithm which, given $n\in\mathbb{N}$, prints $b_n\in\{0,1\}$ where $r=\sum_n b_n 2^{-n}$

a) r has a computable binary expansion

b) There is an algorithm printing, on input $m\in\mathbb{N}$, some $a_m\in\mathbb{Z}$ with $|r-a_m/2^m|\leq 2$

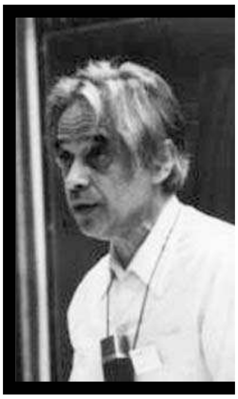
$\Leftrightarrow r\in[q_n\pm\varepsilon_n]$

c) There is an algorithm printing two sequences $(q_n)\subseteq\mathbb{Q}$ and (ε_n) with $|r-q_n|\leq\varepsilon_n\to 0$

numerators+ denominators

b) $\Leftrightarrow$ c) holds *uniformly*, $\Leftrightarrow$ a) does not [Turing'37]

interval arithmetic



Ernst Specker (1949): (c) $\Leftrightarrow$ *Halting problem* plus (d)

d) There is an algorithm printing $(q_n)\subseteq\mathbb{Q}$ with $q_n\to r$.

$H := \{ \langle B,\underline{x}\rangle : \text{algorithm } B \text{ terminates on input } \underline{x} \} \subseteq \mathbb{N}$

Examples: Computable Reals

a) Every dyadic rational has two binary expansions

b) Every rational has a computable binary expansion

c) If a,b are computable, then also $a+b$, $a\cdot b$, $1/a$ $(a\neq 0)$

d) Fix $p\in\mathbb{R}[X]$. Then p's coefficients are computable
$\Leftrightarrow$ $p(x)$ is computable for all computable x.

e) Every algebraic number is computable; and so is π.

f) If x is computable, then so are $\exp(x)$, $\sin(x)$, $\log(x)$

g) Specker's sequence $(\sum_{m<j,\ t(m)<j} 2^{-m})_j$ is computable,
where $\{0,1,2,\ldots,\infty\} \ni t(\langle\mathcal{A},x\rangle)$:=#steps $\mathcal{A}$ makes on x.

$r\in\mathbb{R}$ computable iff an algorithm can print, on input $m\in\mathbb{N}$, some $a\in\mathbb{Z}$ with $|r-a/2^m|\leq 2^{-m}$

Oracle-Computable Real Numbers

Reminder: $r\in\mathbb{R}$ is computable iff some algorithm can print on any input n some $a\in\mathbb{Z}$ s.t. $|r-a/2^n|\leq 2^{-n}$.

Call $(r_j)\subseteq\mathbb{R}$ computable iff an algorithm can print, on input $\langle n,j\rangle\in\mathbb{N}$, some $a\in\mathbb{Z}$ with $|r_j-a/2^n|\leq 2^{-n}$.

Theorem: If computable sequence (r_j) converges, then the real $r:=\lim_j r_j$ is computable relative to H.

And to every real r computable relative to H, there is a computable sequence (r_j) with $r:=\lim_j r_j$.

Lemma: Suppose $(a_m)\subseteq\mathbb{Z}$ satisfies $|r-a_m/2_m|\leq 2^{-m+1}$.
Then $a'_m:=\lfloor a_{m+2}/4\rceil$ satisfies $|r-a'_m/2_m|\leq 2^{-m}$.

$H = \{ \langle\mathcal{B},\underline{x}\rangle$: algorithm $\mathcal{B}$ terminates on input $\underline{x}$ $\} \subseteq\mathbb{N}$

Uniformity: Sequences and Functions

Proposition: If (r_j) is a computable sequence s.t. $|r_j - r_i| \leq 2^{-i} + 2^{-j}$, then $\lim_j r_j$ is a computable real.

Call $(r_j) \subseteq \mathbb{R}$ computable iff an algorithm can print, on input $\langle n,j \rangle \in \mathbb{N}$, some $a \in \mathbb{Z}$ with $|r_j - a/2^n| \leq 2^{-n}$.

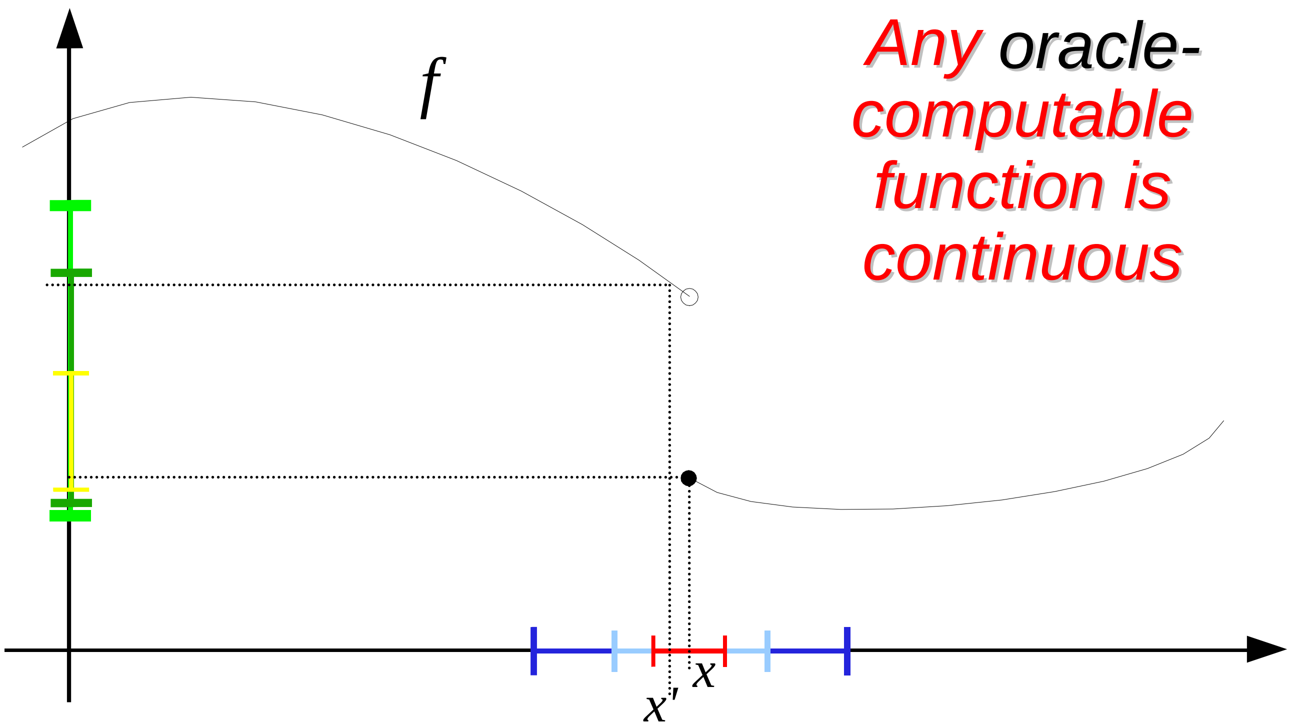
Example of a computable sequence $(r_j) \subseteq [0,1]$ such that $\{ j : r_j \neq 0 \} = H$, the Halting problem.

In numerics, don't test for (in-)equality!

Call $f :\subseteq \mathbb{R} \to \mathbb{R}$ computable iff an algorithm can convert any $(a_m) \subseteq \mathbb{Z}$ with $|x - a_m/2^m| \leq 2^{-m}$ into some $(b_n) \subseteq \mathbb{Z}$ with $|f(x) - b_n/2^n| \leq 2^{-n}$

$H = \{ \langle \mathcal{B}, \underline{x} \rangle$: algorithm $\mathcal{B}$ terminates on input $\underline{x}$ $\} \subseteq \mathbb{N}$

Uniformly Computable Real Functions



Any oracle-computable function is continuous

$x \in \mathbb{R}$ computable $\Leftrightarrow$ $|x - a_n/2^n| \leq 2^{-n}$ for recursive $(a_n) \subseteq \mathbb{Z}$

Computable Weierstrass Theorem

KAIST
CS700 M. Ziegler

Theorem: For $f:[0,1]\to\mathbb{R}$ the following are equivalent:

a) There is a machine <u>converting</u> any $\underline{q}=(q_m)$, $q_m\in\mathbb{D}_n$ with $|x-q_m|\leq 2^{-m}$, into $(p_n)\in\mathbb{D}_n$ with $|f(x)-p_n|\leq 2^{-n}$

b) There is a machine <u>printing</u> a sequence (of degrees and coefficient lists of) $(P_n)\subseteq\mathbb{D}[X]$ with $\|f-P_n\|_\infty\leq 2^{-n}$

c) The real sequence $f(q)$, $q\in\mathbb{D}\cap[0,1]$, is computable $\wedge$ f admits a computable modulus of (unif) continuity

$|x-y|\leq 2^{-\mu(n)} \Rightarrow |f(x)-f(y)|\leq 2^{-n}$

Proof: b $\Leftrightarrow$ c $\Rightarrow$ a $\Rightarrow$ c

Call $(r_j)\subseteq\mathbb{R}$ computable iff an algorithm can print, on input $n,j\in\mathbb{N}$, some $q\in\mathbb{D}_n$ with $|r_j-q|\leq 2^{-n}$.

$\mathbb{D}:=\bigcup_n \mathbb{D}_n$, $\qquad \mathbb{D}_n:=\{\,a/2^n : a\in\mathbb{Z}\,\}$

Compactness in Real Computation

KAIST
CS700 M. Ziegler

Lemma: Let machine $\mathcal{A}$ convert any $\underline{a}=(a_m)\subseteq\mathbb{Z}$ s.t. $|x-a_m/2^m|\leq 2^{-m}$, $x\in[0;1]$, to (b_n) s.t. $|f(x)-b_n/2^n|\leq 2^{-n}$.

a) $t_{\mathcal{A}}(n):\underline{a}\to$#steps $\mathcal{A}$ makes on input $\underline{a}$ to print b_n is locally constant (=<u>continuous</u>) a function

b) giving rise to a *modulus of <u>local</u> continuity* to f:

$$\forall x\ \exists\underline{a}:\quad |x-x'|\leq 2^{-t(n,\underline{a})-1} \quad\Rightarrow\quad |f(x)-f(x')|\leq 2^{-n+1}$$

c) Its domain $\{\,\underline{a}\in\mathbb{Z}^{\mathbb{N}}: \exists x\in[0;1]\ \forall m: |x-a_m/2^m|\leq 2^{-m}\,\}$ is <u>compact</u> in Baire Space $\mathbb{Z}^{\mathbb{N}}$ wrt $d(\underline{a},\underline{b})=2^{-\min\{n:a_n\neq b_n\}}$

d) and its set of finite initial segments is <u>co-r.e.</u> =
$\{\bar{a}\in\mathbb{Z}^{m}$: $m\in\mathbb{N}$, $\forall i,j$: $-1\leq a_j\leq 1+2^j \wedge |a_i/2^i - a_j/2^j|\leq 2^{-i}+2^{-j}\,\}$

e) $t_{\mathcal{A}}:\mathbb{N}\ni n\to\max_{\underline{a}} t_{\mathcal{A}}(n,\underline{a})$ is well-def. and <u>recursive</u>

Compactness in Baire Space

Complexity gauge: discrete $t_{\mathcal{A}}(\underline{x})$, $\underline{x}\in\{\mathbf{0},\mathbf{1}\}^*$

$$t_{\mathcal{A}}(n) := \max\{\, t_{\mathcal{A}}(\underline{x}) : |\underline{x}|\le n \,\}$$

real arguments: $t_{\mathcal{A}}(\underline{a},n)$,

$$t_{\mathcal{A}}(n) := \max\{\, t_{\mathcal{A}}(\underline{a},n) : \forall m: |x-a_m/2^m|\le 2^{-m},\ x\in[0;1]\,\}$$

König's Lemma: $X\subseteq\mathbb{Z}^{\mathbb{N}}$ is compact iff it is closed and the set $X^* := \{\, \bar{a}\in\mathbb{Z}^* \mid \exists\underline{b}\in\mathbb{Z}^{\mathbb{N}} : \bar{a}\underline{b}\in X \,\}$ of finite initial segments is finitely branching.

(X,d) *compact* iff every sequence has a converging sub-sequence.

$\mathbb{Z}^{\mathbb{N}}$ wrt $d(\underline{a},\underline{b})=2^{-\min\{n:a_n\neq b_n\}}$ is *not* compact: $(\,(0,0,\ldots)\,,\,(1,1,\ldots),\,(2,2,\ldots)\,,\,\ldots\,)$

Examples of Computable Real Functions

a) f computable $\Rightarrow$ so is any restriction of f

b) $\exp$, $\sin$, $\cos$, $\ln(1+x)$ are computable functions

c) For a computable sequence $\underline{a}=(a_n)$, the power series $x\to\sum_n a_n\cdot x^n$ is computable on $(-r,r)$ for fixed $r < R(\underline{a}) := 1/\limsup_n |a_n|^{1/n}$

d) Let $f\in C[0,1]$ be computable. Then so are $\int f: x\to\int_0^x f(t)\,dt$ and $\max(f):x\to\max\{f(t):t\le x\}$.

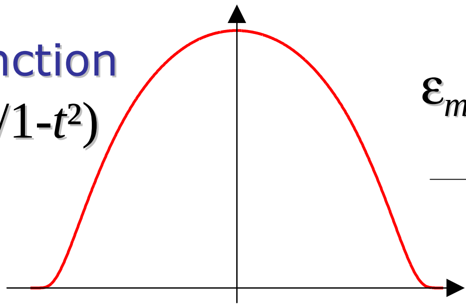
e) If $(x,m)\to f_m(x)$ computable with $\|f_n-f_m\|_\infty\le 2^{-n}+2^{-m}$ then $\lim_n f_n$ is again computable.

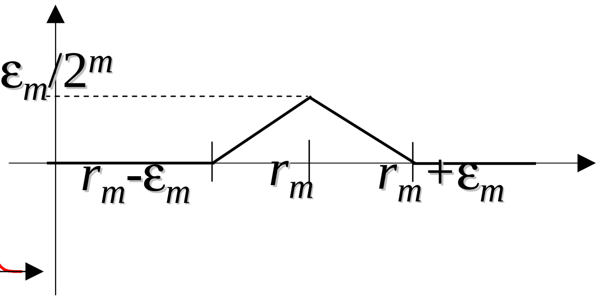
uncomputable

f) For computable $a\in\mathbb{R}$, $f:[0,a]\to\mathbb{R}$, and $g:[a,1]\to\mathbb{R}$ with $f(a)=g(a)$, their join is computable

Computable Urysohn

C$^\infty$ 'pulse' function
$\varphi(t) = \exp(-t^2/1-t^2)$
$|t|<1$





Let $(r_m)_m, (\varepsilon_m)_m \subseteq \mathbb{Q}$ be computable sequences
Then there is a computable C^∞ $f:[0;1]\to[0;1]$
s.t. $f^{-1}[0] = [0;1]\setminus\bigcup_m (r_m-\varepsilon_m, r_m+\varepsilon_m)$.

Proof: Let $f(x) := \sum_m \max(0, \varepsilon_m - |x-r_m|)/2^m$

Specker'59: Uncomputable `argmin`

Lemma: There are computable sequences $(r_m)_m, (\varepsilon_m)_m \subseteq \mathbb{Q}$ s.t. $U := \bigcup_m (r_m-\varepsilon_m, r_m+\varepsilon_m)$ contains all computable reals in [0;1] and has measure $\le$½.

approximating a root
vs. approximate root

Let $(r_m)_m, (\varepsilon_m)_m \subseteq \mathbb{Q}$ be computable sequences
Then there is a computable C^∞ $f:[0;1]\to[0;1]$
s.t. $f^{-1}[0] = [0;1]\setminus\bigcup_m (r_m-\varepsilon_m, r_m+\varepsilon_m)$.

Corollary: There is a computable C$^\infty$ $f:[0;1]\to[0;1]$ s.t. $f^{-1}[0]$ has measure $\ge$½ but contains no computable real number.

Singular Covering of Computable Reals

Lemma: There are computable sequences $(r_m)_m, (\varepsilon_m)_m \subseteq \mathbb{Q}$ s.t. $U := \bigcup_m (r_m-\varepsilon_m, r_m+\varepsilon_m)$ contains all computable reals in [0;1] and has measure ≤½.

$\mathcal{A}$ computes $r\in\mathbb{R}$ iff prints sequence $a_n\subseteq\mathbb{Z}$ with $|a_n/2^n - a_m/2^m| \leq 2^{-n} + 2^{-m}$

Proof idea (diagonalize against <u>all</u> $\mathcal{A}$):

Simulate machine $\mathcal{A}$ until it outputs $(a_0,a_1,\ldots a_{\langle\mathcal{A}\rangle+4})\in\mathbb{Z}^*$

s.t. $0\leq a_n\leq 2^n$, $|a_n/2^n-a_m/2^m| \leq 2^{-n}+2^{-m}$ $\forall n,m\leq\langle\mathcal{A}\rangle+4$

and let $r_{\langle\mathcal{A}\rangle} := a_{\langle\mathcal{A}\rangle+4}/2^{\langle\mathcal{A}\rangle+4}$ and $\varepsilon_{\langle\mathcal{A}\rangle} := 2^{-\langle\mathcal{A}\rangle-3}$.

Then $\lambda(U) \leq \sum_{\langle\mathcal{A}\rangle} 2\varepsilon_{\langle\mathcal{A}\rangle} = ½$ and $\mathbb{R}_c\subseteq U$.

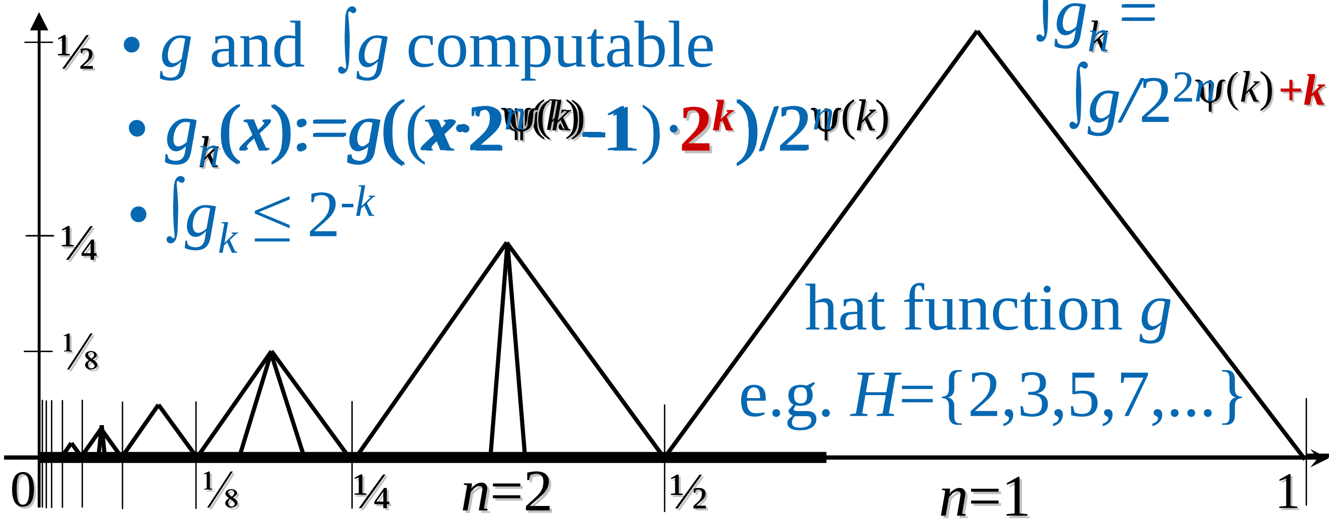
What if $\mathcal{A}$ does not produce infinite output?

Myhill'71: uncomputable ∂ on $C^1[0,1]$

Recall computable bijection $\psi:\mathbb{N}\to H$

- g and $\int g$ computable
- $g_k(x) := g((x\cdot 2^{\psi(k)}-1)\cdot 2^k)/2^{\psi(k)}$
- $\int g_k \leq 2^{-k}$

$\int g_k = \int g/2^{2\psi(k)+k}$



$h' := \sum_{k\in\mathbb{N}} g_k$ continuous, incomputable,

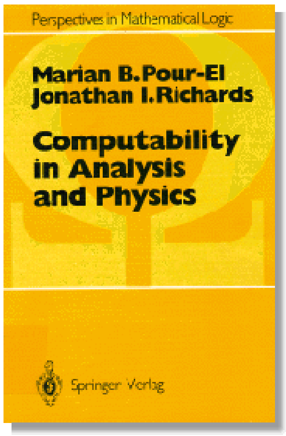
yet $h := \int h' \in C^1[0;1]$ computable. q.e.d.

The Case of the Wave Equation

KAIST
CS700 M. Ziegler

Myhill'71: computable $h \in C^1[0,1]$ with uncomputable $h'(1)$

Pour-El&Richards'81 construct a computable $f \in C^1(\mathbb{R}^3)$ such that for g:=0 the unique solution is *in*computable at t=1 and $\underline{x}$=(0,0,0).



Church-Turing Hypothesis (Kleene):
Everything that can be computed by a Turing machine can also be computed by a physical device – and vice versa!

$\partial^2/\partial t^2\, u(\underline{x},t) = \Delta u(\underline{x},t), \quad u(\underline{x},0)=f(\underline{x}), \quad \partial/\partial t\, u(\underline{x},0)=g(\underline{x})$

The Case of the Wave Equation

KAIST
CS700 M. Ziegler

Myhill'71: computable $h \in C^1[0,1]$ with uncomputable $h'(1)$

Pour-El&Richards'81 construct a computable $f \in C^1(\mathbb{R}^3)$ such that for g:=0 the unique solution is *in*computable.

Kirchhoff's formula:

$$u(t,\vec{x}) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\vec{y}-\vec{x}|=t} f(\vec{y})\, d\sigma(\vec{y})\right) + \frac{1}{4\pi t}\int_{|\vec{y}-\vec{x}|=t} g(\vec{y})\, d\sigma(\vec{y})$$

$$f(\vec{x}) := h(|\vec{x}|^2)$$

$$u(t,0) = \frac{d}{dt}\Big(h(t^2)\cdot t\Big) = h'(t^2)\cdot 2t^2 + h(t^2)$$

$\partial^2/\partial t^2\, u(\underline{x},t) = \Delta u(\underline{x},t), \quad u(\underline{x},0)=f(\underline{x}), \quad \partial/\partial t\, u(\underline{x},0)=g(\underline{x})$